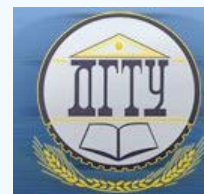


## MECHANICS



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**Infinite plate loaded with normal force moving along a complex path****A. V. Galaburdin**

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*Introduction.* A technique of solving the problem on an infinite plate lying on an elastic base and periodically loaded with a force that moves along an arbitrary closed trajectory and according to an arbitrary law, is considered.

*Materials and Methods.* An original method for solving problems on the elasticity theory for plates loaded with a force moving arbitrarily along a closed trajectory of arbitrary shape is considered. The problem on an infinite plate lying on an elastic foundation is investigated. The plate is loaded with a normal force moving at a variable speed. The load is decomposed into a Fourier series on a time interval whose length is equal to the time of its passage along the trajectory. The solution to this problem is realized through a superposition of solutions to the problems corresponding to the load defined by the summands of the specified Fourier series. The final problem solution is presented in the form of a segment of the Fourier series, each summand of which corresponds to the solution to the problem on the action on an infinite plate of the load distributed along a closed trajectory of the force motion. The fundamental solution to the vibration equation of an infinite plate lying on an elastic foundation is used to construct these solutions.

*Results.* A solution to the problem of an infinite plane, along which a concentrated force moves at a variable speed, is presented. A smooth closed curve consisting of arcs of circles was considered as a trajectory. The behavior of displacements and stresses near the moving force is investigated; and the process of the elastic wave energy propagation is also studied. For this purpose, a change in the Umov-Poynting vector is considered.

*Discussion and Conclusions.* The results obtained can be used in calculations for road design. The study of the propagation of the energy of elastic waves from moving vehicles will provide the assessment of the impact of these waves on buildings located near the road. Analysis of the behavior of displacements and stresses near the moving force will allow assessing the wear of the road surface.

**Keywords:** infinite plate, moving load, arbitrary closed trajectory, variable speed, elastic wave energy.

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**Introduction.** In many areas of science and technology, problems related to the propagation of elastic waves arise. This work objection is to study laws of the propagation of elastic waves that occur under the action of a moving load. Problems of this kind were previously investigated by various authors in a number of papers, which considered a variety of problem statements with a moving load and proposed various solution methods.

Often, when solving such problems, a mobile coordinate system is introduced [1–5], or a quasi-static statement of the problem is considered [6–12] to exclude time from the number of independent variables. A number of papers use the finite element method [11–13]. Interesting results can be obtained using variational [14–16] or direct techniques

[17–19]. Methods based on the application of fundamental solutions to the corresponding differential equations were used in [20–22] under solving problems on the elasticity theory about a force moving at a constant speed.

**Problem Setting.** Consider a differential equation describing vibration of an infinite plate lying on an elastic base under the action of vertical force  $P$ :

$$\Delta^2 U + c^{-2} \partial_t^2 U + kU = \frac{P}{D}, \quad (1)$$

where  $U$  is deflection of the plate;  $D = \frac{Ed^3}{12(1-\mu^2)}$ ;  $E$  is Young's modulus;  $\mu$  is Poisson's ratio;  $d$  is thickness of the plate;  $c^{-2} = \frac{\rho d}{D}$ ;  $\rho$  is density of the plate material;  $k = \frac{K}{D}$ ;  $K$  is coefficient of rigidity of the elastic base.

We will consider a solution for which the energy flow is directed from the sources of elastic wave excitation to infinity, and assume that force  $P$  moves along a closed trajectory  $\gamma$  in an arbitrary way. In addition, we assume that  $P = P(s(t))$ , where  $s$  is the arc coordinate counted from some fixed point of the curve  $\gamma$ . Obviously,  $P = P(s(t))$  will be a periodic function over  $t$  with period  $T$ , if  $s(t)$  is also a periodic function over  $t$  with period  $T$ .

**Materials and Methods.** Consider the fundamental solution to the equation (1). It can be obtained from the expression:

$$\Delta^2 W + c^{-2} \partial_t^2 W + kW = \frac{1}{D} \delta(x - x_o) \delta(y - y_o) \delta_T(t - \tau), \quad (2)$$

where  $\delta_T(t - \tau) = \sum_{n=-\infty}^{\infty} \delta(t - \tau - nT)$ .

The solution to the equation (2) can be obtained by traditional methods, using the limiting absorption principle, and presented as a series:

$$W(x, x_o, y, y_o, t - \tau) = \sum_{k=-\infty}^{\infty} w_k(x, x_o, y, y_o, \omega_k) e^{-i\omega_k(t-\tau)}, \quad (3)$$

where  $w_k(x, x_o, y, y_o, \omega_k)$  satisfies the equation:

$$\Delta^2 w_k - c^{-2} \omega_k^2 w_k + kw_k = \frac{1}{D} \delta(x - x_o) \delta(y - y_o) e^{i\omega_k \tau}.$$

It is known that the solution to the equation (1) can be presented as:

$$U(x, y, t) = \frac{1}{T} \int_{-T/2}^{T/2} \iint_{R^2} W(x, x_o, y, y_o, t - \tau) P(x_o, y_o, \tau) dx_o dy_o d\tau.$$

If the moving force is a single concentrated force, which is described by the function:

$$P(s(t)) = \delta(x - x_1(s(t))) \delta(y - y_1(s(t))),$$

the solution in this case will look like:

$$U(x, y, t) = \frac{1}{T} \int_{-T/2}^{T/2} W(x, x_1(s(\tau)), y, y_1(s(\tau)), t - \tau) d\tau. \quad (4)$$

Once defined  $U(x, y, t)$ , it is possible to calculate displacements and stresses at any point of the plate.

If the expression  $\omega_k = \frac{2k\pi}{T}$  is large enough, it is required to calculate integrals of rapidly oscillating functions

(3). For this purpose, the quadrature formula was used [23, 24]:

$$\int_a^b e^{i\omega x} f(x) dx \approx \int_a^b e^{i\omega x} S(x) dx \approx -\frac{1}{\omega^4} \sum_{j=1}^{N-1} \frac{e^{i\omega x_{j+1}} - e^{i\omega x_j}}{h_j} (M_{j+1} - M_j),$$

where  $h_j$  are the lengths of elementary segments into which the interval  $[a; b]$  is divided;  $S(x)$  is approximation of  $f(x)$  by cubic spline  $M_j = S''(x_j)$ .

**Research Results.** The calculations were performed for an infinite plate loaded with a normal force that moved along a closed curve (Fig. 1).

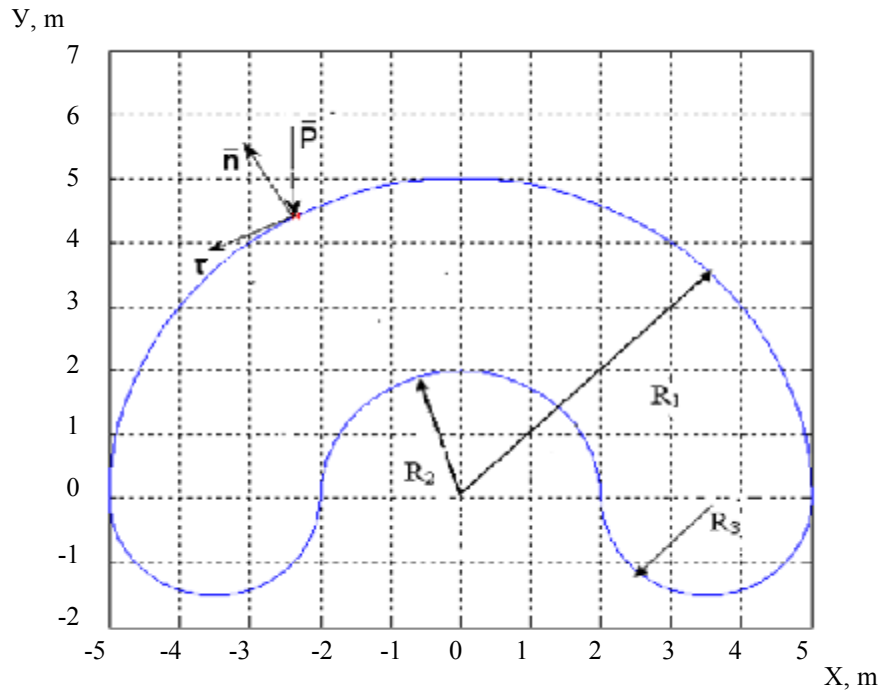


Fig. 1. Trajectory of the concentrated force

During the calculations, it was assumed that the plate thickness was  $d = 0.25$  m, parameter  $c = 221$  m/s, young's modulus of the plate material  $E = 232469$  H/m<sup>2</sup>, Poisson's ratio  $\mu = 0.36$ , elastic base compliance coefficient  $K = 1.864$  m<sup>-4</sup>. The radii that determine the shape of the force trajectory were assumed to be equal to:  $R_1 = 5$  m,  $R_2 = 3$  m,  $R_3 = 1$  m. The formula (3) held 120 terms of the Fourier series, and, when calculating the integral (4), the integration interval was divided into 120 equal subintervals.

The law of force motion along the trajectory was described by the function:

$$s(t) = \frac{L \cdot \sin(\alpha(t - T/2))}{2 \sin(\alpha T/2)} + \frac{L}{2}, \text{ где } \alpha = \frac{\pi}{T}, t \in [0 : T].$$

We considered the instant of time  $t = \frac{T}{2}$ , when the moving force was at the same point of the trajectory for any

$T$ . When the parameter  $T$  is changed, the speed rate of the concentrated force along the trajectory changes.

To analyze the stress-strain state of the plate, displacements and stresses were calculated in a rectangular coordinate system associated with a moving concentrated force. In this case, the axis  $\tau$  of this system was directed tangentially to the trajectory  $\bar{\tau}$ , and  $n$  axis coincided in the direction of the external normal to the area bounded by the trajectory  $\bar{n}$  (Fig. 1).

In this coordinate system, the displacement vector and stress tensor can be presented as:

$$\bar{U} = U\tau \cdot \bar{\tau} + Un \cdot \bar{n} + W \cdot \bar{k},$$

$$\bar{\bar{S}} = St \cdot \bar{\tau}\bar{\tau} + Sn \cdot \bar{n}\bar{n} + Stn \cdot (\bar{\tau}\bar{n} + \bar{n}\bar{\tau}),$$

where  $\bar{k}$  is the normal to the plate.

Fig. 2 shows the variation of the components of the displacement vector  $Wt, Utt, Unt$ , the stress tensor  $Stt, Snt, Stnt$  (second and third graphs) in the axis  $\tau$ , and the change in the same values along the axis  $n$  —  $Wn, Unt, Unn$  and  $Stn, Snn, Stnn$  (first and fourth graphs) at  $z = h/2$ , the speed of the force  $v = 49.3480$  m/s, acceleration along the trajectory  $w_t = 0$  m/s<sup>2</sup>, and normal acceleration  $w_n = v^2/R_1 = 487.045$  m/s<sup>2</sup> (the position of the force on the trajectory is marked with a red dot in Fig. 1).

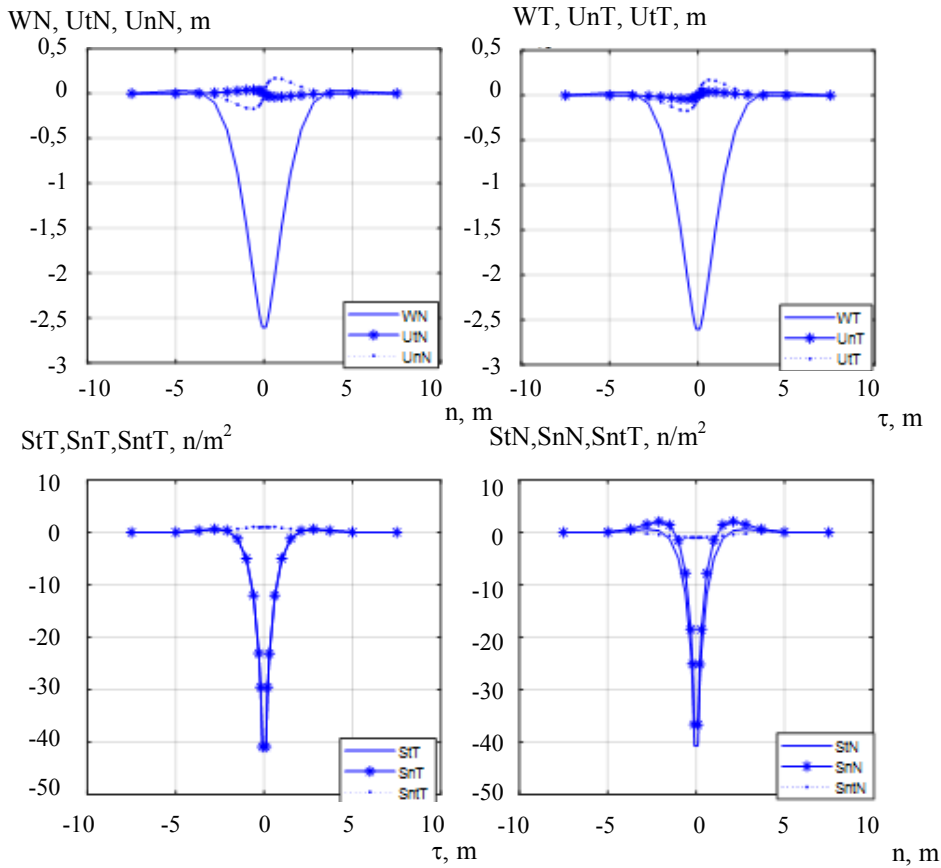


Fig. 2. Variation of displacements and stresses ( $v = 49.3480$  m/s)

When studying the energy propagation of elastic waves, the energy flux vector was calculated:

$$\vec{P} = -(s_{11}\dot{u}_1 + s_{12}\dot{u}_2)\vec{i} - (s_{12}\dot{u}_1 + s_{22}\dot{u}_2)\vec{j},$$

where  $s_{ij}$  are components of the stress tensor;  $\dot{u}_i$  is time derivative of the coordinates of the displacement vector.

Fig. 3 shows the propagation of elastic wave energy near a concentrated force whose position on the trajectory is indicated with a red dot. The length of the vector shown corresponds to the amount of energy passing through a given point in space per unit of time, and the direction of the vector indicates the direction of energy transfer.

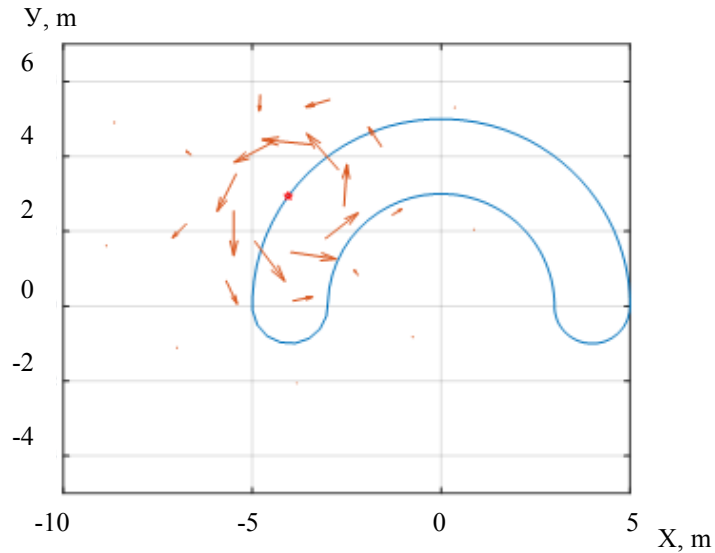


Fig. 3 Energy flux vector ( $v = 49.3480$  m/s)

The calculations have shown that with an increase in the force motion speed, the behavior of displacements and stresses, as well as the nature of the elastic wave energy propagation, changes slightly. Fig. 4 and 5 show the calculation results for speed  $v = 246.7401$  m/s, which exceeds the elastic wave velocity in the plate —  $c = 221$  m/s. The effect of the concentrated force speed on the distribution of vertical displacements  $W$  is shown in Fig. 6, 7.

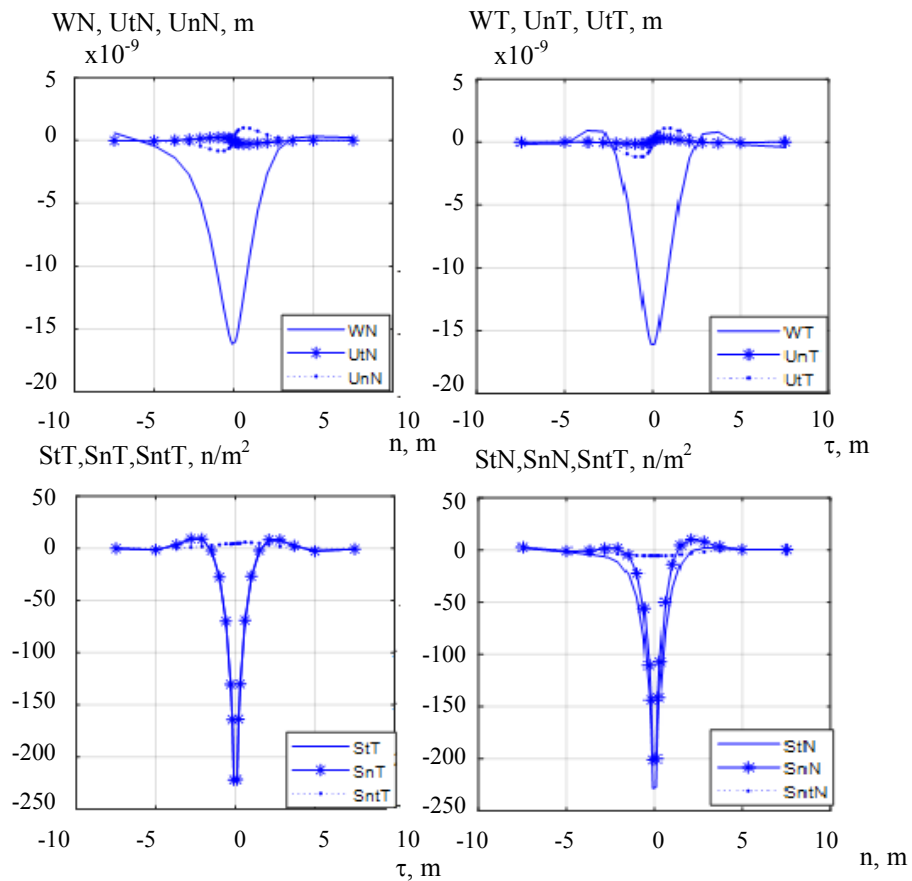


Fig. 4. Variation of displacements and stresses ( $v = 246.7401$  m/s)

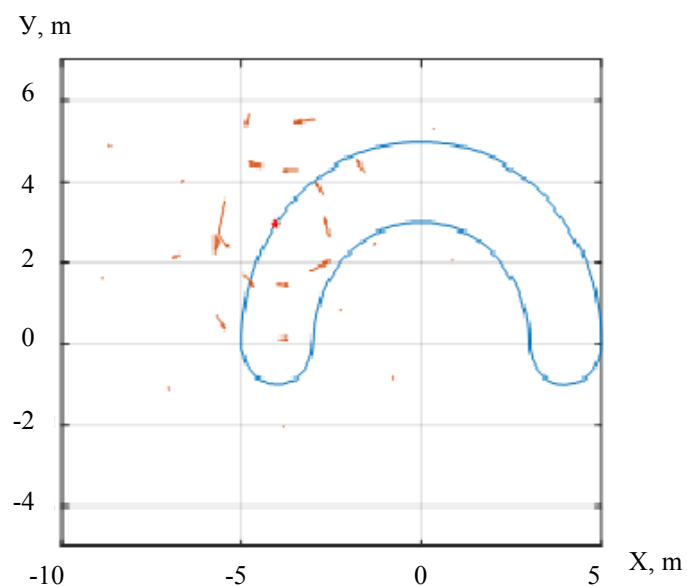


Fig. 5. Energy flux vector ( $v = 246.7401$  m/s)

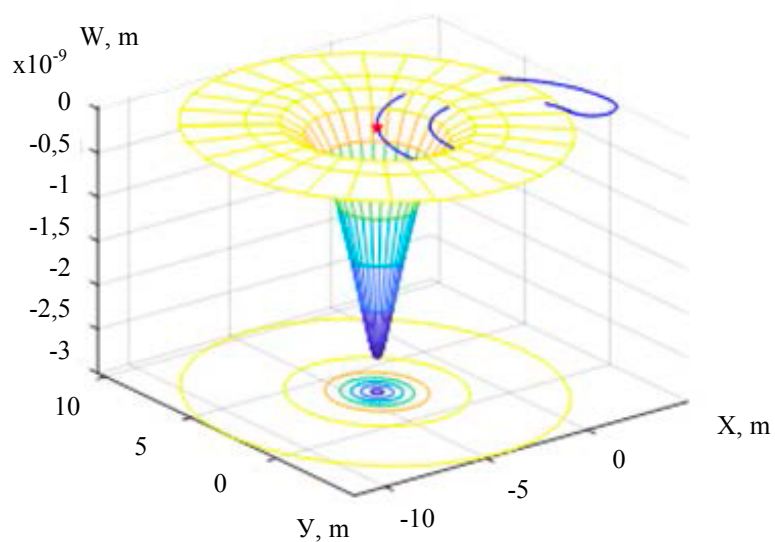


Fig. 6. Variation of vertical displacements at speed  $v = 49.3480$  m/s

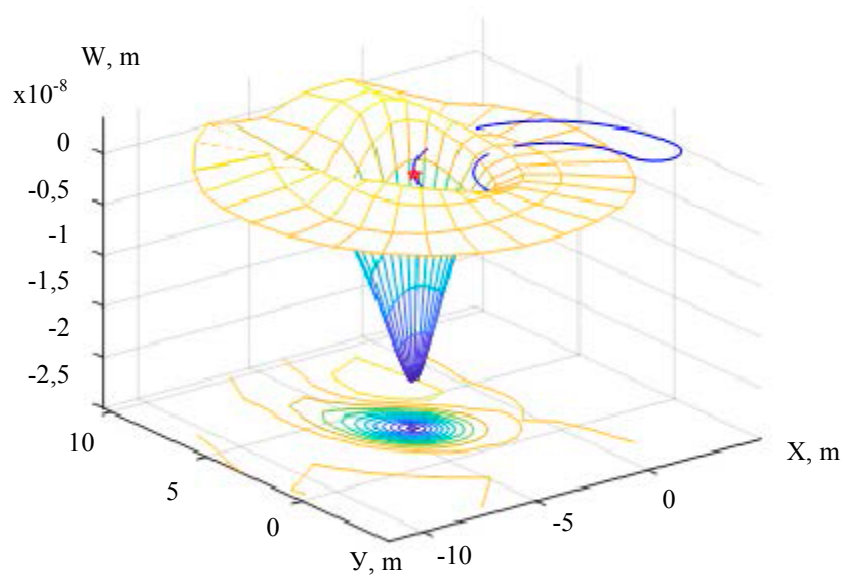


Fig. 7. Variation of vertical displacements at speed  $v = 493.480$  m/s

Fig. 8 and 9 show peak value graphs of displacements and stresses depending on the concentrated force motion speed. The position of the force on the trajectory at the time under consideration is marked with a dot in Fig. 3 and 5.

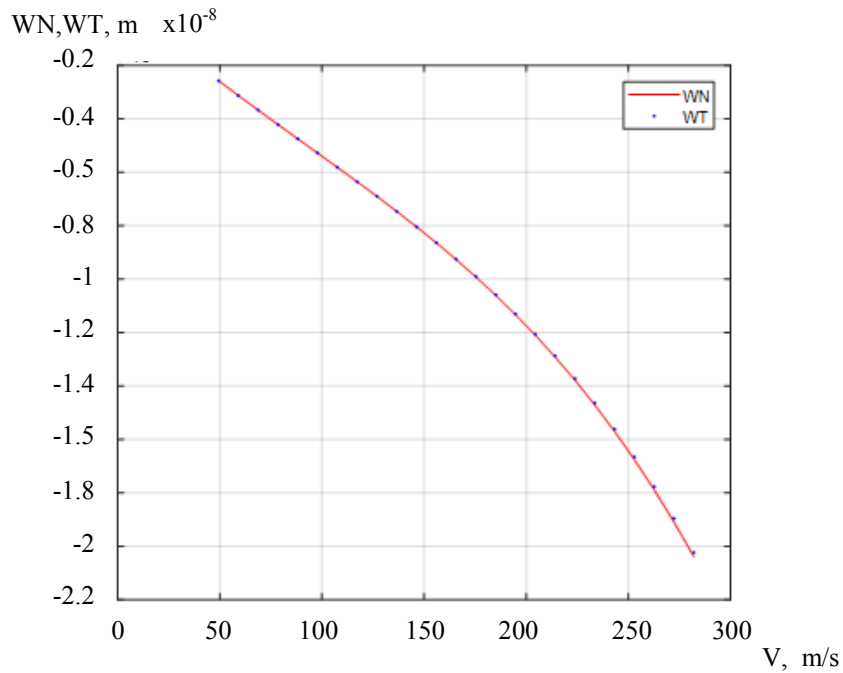


Fig. 8. Change in maximum displacements depending on the concentrated force motion speed

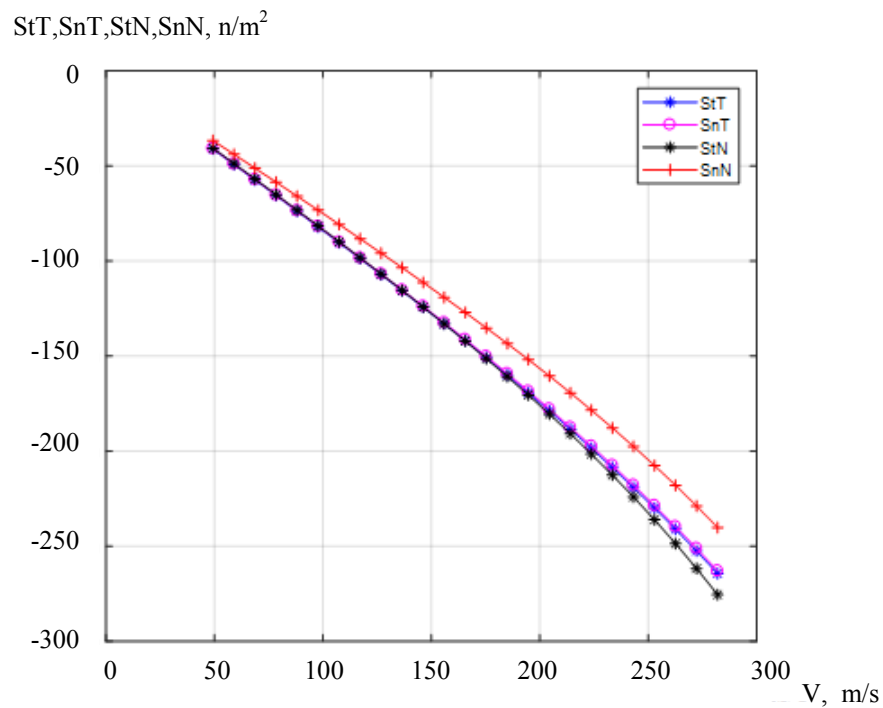


Fig. 9. Change in maximum voltage values depending on the concentrated force speed

Fig. 10 shows the elastic wave energy propagation near a moving concentrated force. The tangential acceleration of the moving force at this moment was equal  $w_t = 1.5503 \text{ m/s}^2$ . The calculations are performed for the instant of time  $t = T$  with the same law of force motion along the trajectory as in the previous case, hence with the same law of change in the speed and acceleration of the force motion.

At this instant in time, the force was at the point of the trajectory shown in Fig. 10, and its speed was zero. Fig. 11 shows the variation of components of the displacement vector  $W_t, U_{tt}, U_{nt}$  and the stress tensor  $Stt, Snt, Stnt$  in  $t$  axis.

An increase in the force acceleration also caused changes in displacements and stresses and affected the nature of elastic wave energy propagation. Fig. 12 and 13 show the calculation results for the case  $w_t = -155.0314 \text{ m/s}^2$ .

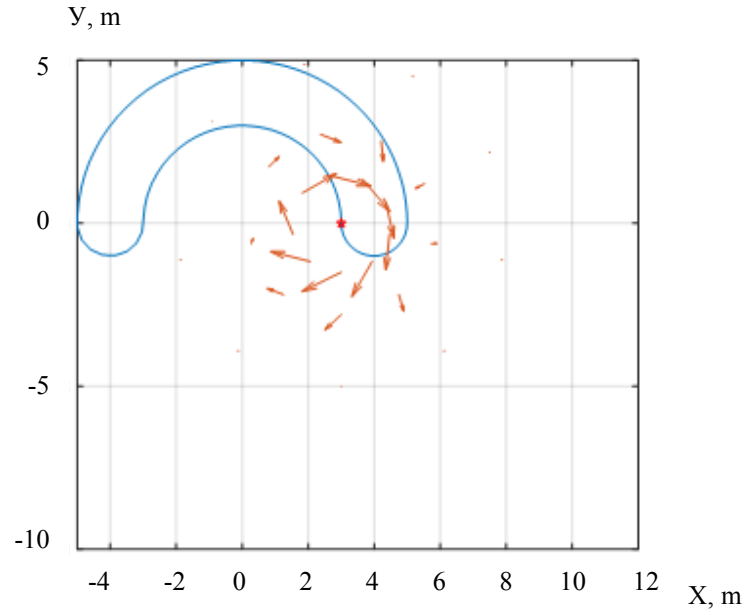


Fig.10 Energy flux vector ( $w_t = 1.5503 \text{ m/s}^2$ )

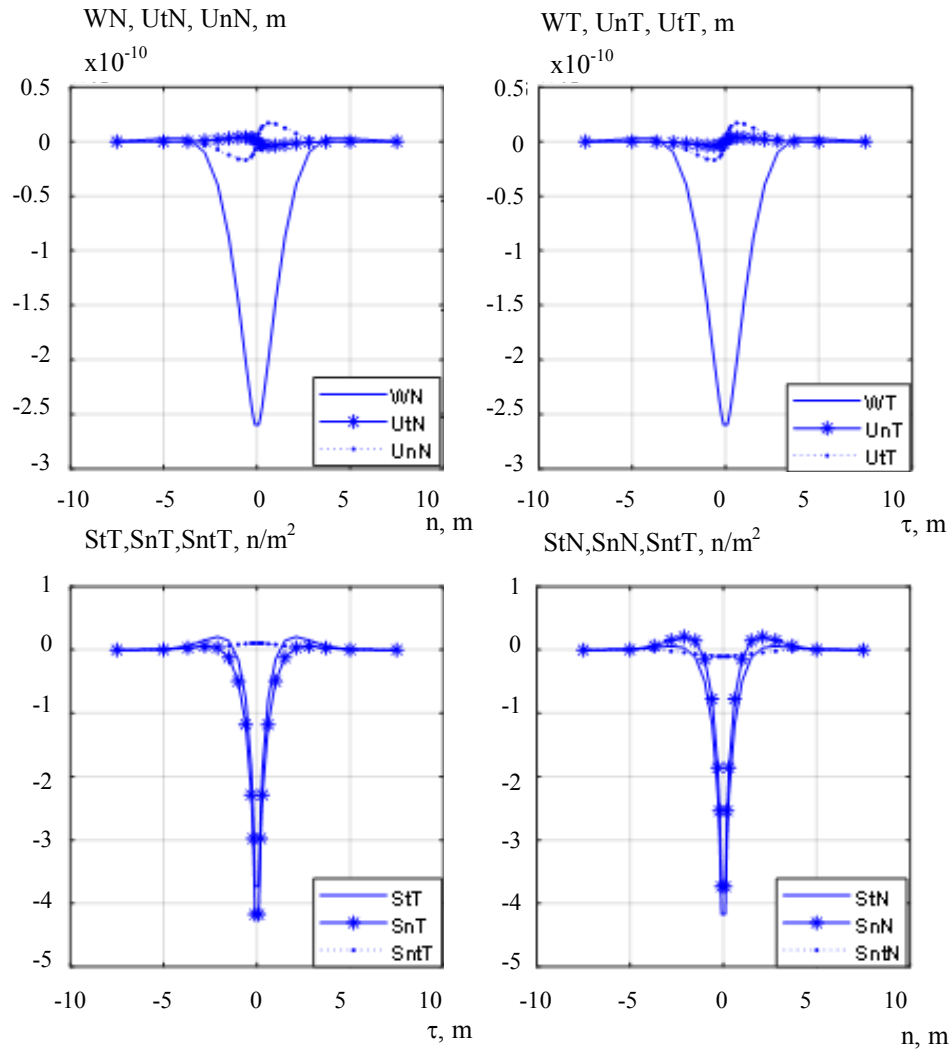


Fig. 11. Variation of displacements and stresses ( $w_t = 1.5503 \text{ m/s}^2$ )

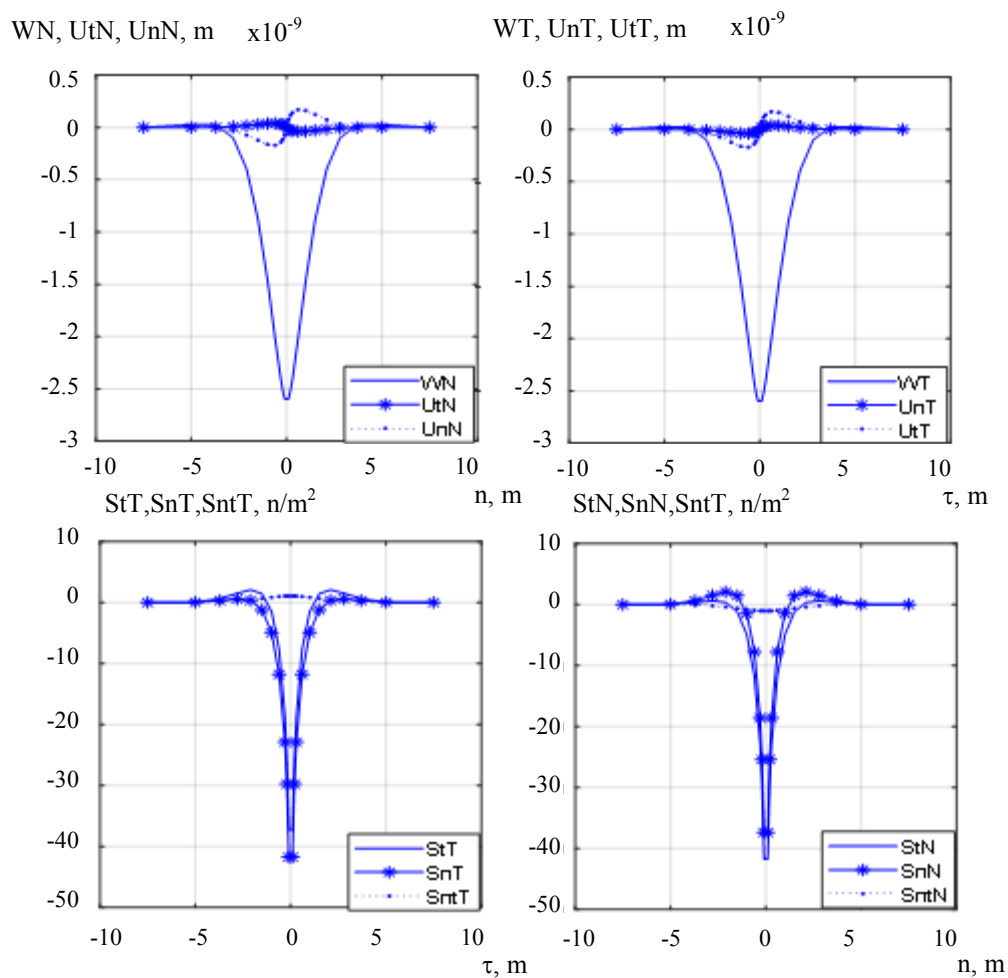


Fig. 12. Variation of displacements and stresses ( $w_t = 155.0314 \text{ m/s}^2$ )

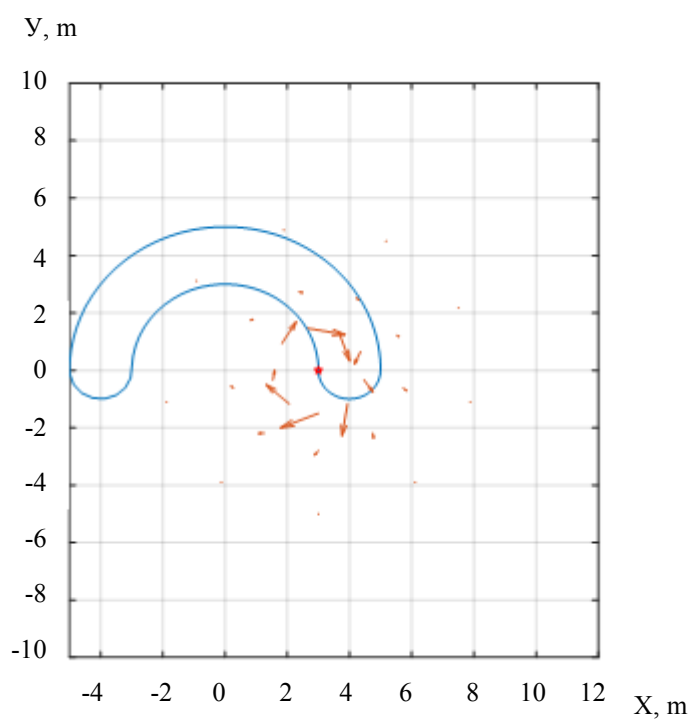


Fig. 13. Energy flux vector ( $w_t = 155.0314 \text{ m/s}^2$ )

Fig. 14 and 15 show peak value graphs of displacements and stresses depending on the tangential acceleration of the concentrated force.

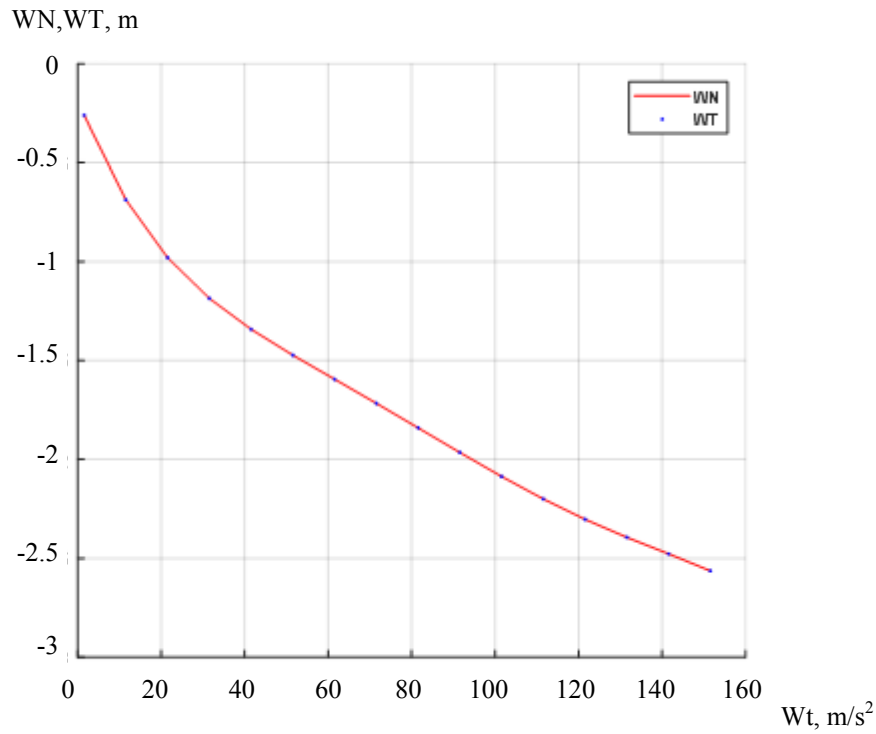


Fig. 14. Peak value graph of displacements depending on tangential acceleration

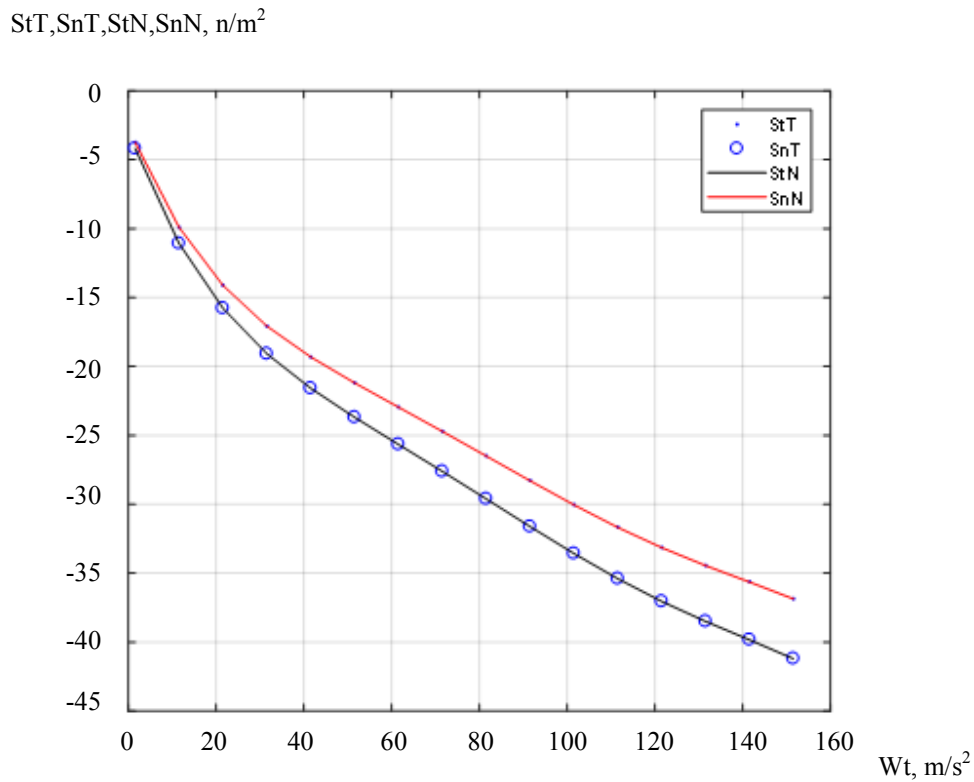


Fig. 15. Peak value graph of stress values depending on tangential acceleration

**Discussion and Conclusions.** There is a pronounced dependence of the fields of displacements and stresses on the speed and acceleration of the force movement at the parameter variation limits considered above. The nature of elastic wave energy propagation also depends significantly on the speed and acceleration.

Sufficiently large values of the speed and acceleration of the force motion were specially considered to test the proposed method under such conditions. The results obtained allow us to conclude that the method is quite stable over a wide range of variable parameters.

The use of the proposed method is quite acceptable for solving more complex problems. For that, it is required that the differential equations describing them provide the analytical construction of the fundamental solution. This method is economical and simple since it uses already known problem decisions to build a solution.

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*The author has read and approved the final manuscript*